TRANSIENT-STATE FLOW OF A CONDUCTING LIQUID IN AN MHD GENERATOR AT CONSTANT FLOW RATE IN THE PRESENCE OF SIDE WALLS

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It is usual in studies of transient [nonsteady] flow for a viscous incompressible conducting fluid in an MHD channel to take the distance between the side walls as infinite, which allows the initial equations to be simplified, these reducing to a single equation for the velocity if the magnetic Reynolds number is small [1–3]. A real system has a finite ratio of the sides, so it is desirable to establish the effects of the side walls.

Consider the transient-state flow at constant flow rate with an arbitrary load coefficient, on the assumption $R_m \ll 1$. This corresponds to adjustment of the output of an MHD generator by alteration of the magnetic induction.

We assume that the device which drives the liquid has a fixed characteristic $Q = f(p)$, which allows the flow rate Q to be kept constant as the pressure p varies.

The equations of magnetic hydrodynamics may be put as

$$\rho\partial \mathbf{v}/\partial t + \rho(\mathbf{v}\nabla)\mathbf{v} = -\nabla p + \eta\Delta\mathbf{v} + [\mathbf{j}\times\mathbf{B}],$$

$$\mathbf{j} = \sigma\{\mathbf{E} + [\mathbf{v}\times\mathbf{B}]\}, \qquad \operatorname{rot}\mathbf{E} = -\partial\mathbf{B}/\partial t,$$

$$\mathbf{j} = \mu^{-1}\operatorname{rot}\mathbf{B}, \quad \operatorname{div}\mathbf{B} = \operatorname{div}\mathbf{v} = 0, \tag{1}$$

in which v is the flow velocity, B is the magnetic induction, E is the electric field, j is current density, and ρ, η, σ, and μ are, respectively, the density, dynamic viscosity, conductivity, and permeability.

The long channel is of rectangular cross section, the sides $x = \pm b/2$ being the nonconducting poles of the magnet, while the sides $y = \pm a/2$ are conducting electrodes joined through the load resistance r (figure); here $a \gg b$, but a is finite. Then Eqs. (1) become

$$\rho\frac{\partial v_z}{\partial t} = -\frac{\partial p}{\partial z} + \eta\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2}\right) - j_y B_x,$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t},$$

$$j_x = \sigma E_x, \quad j_y = -\mu^{-1}\partial B_z/\partial x = \sigma(E_y + v_z B_x). \tag{2}$$

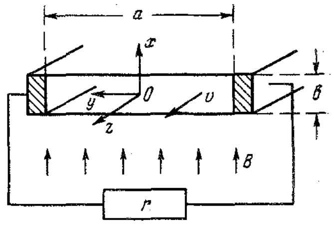
If $a \gg b$ we may put $\partial E_y/\partial x \gg \partial E_x/\partial y$. Further, the orders of magnitude of the other terms in the second equation in (2) are as follows:

$$\frac{\partial E_y}{\partial x} \sim \frac{\partial B_z}{\partial t} \sim R_m\frac{B_x}{T}, \quad \left|\frac{\partial E_y}{\partial x}b\right| : |v_z B_x| \approx R_m\frac{b}{UT},$$

$$R_m = \mu\sigma U R_h, \quad R_h = ab/(a+b),$$

in which R_m is the magnetic Reynolds number, R_h is the hydraulic radius of the channel, U is the mean flow speed, and T is the characteristic time. If $b/UT \lesssim 1$, we may assume that E_y is independent of x and is a function of time alone.

For steady-state problems and $a \gg b$, we may make the approximation E_y = constant, which agrees with experiment [4].



Then system (2), with Ohm's law applied to the external circuit, reduces to one equation for the velocity:*

$$\frac{\partial v}{\partial t} = P(t) + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + k\frac{M^2\nu}{R_h^2}U - \frac{M^2\nu}{R_h^2}v,$$

$$P(t) = -\frac{1}{\rho}\frac{\partial p}{\partial z},$$

$$U = \frac{Q}{ab} = \frac{1}{ab}\int_{-1/2b}^{1/2b}\int_{-1/2a}^{1/2a} v\,dx\,dy = \text{const}, \quad \nu = \frac{\eta}{\rho},$$

$$M = R_h B_x\sqrt{\sigma/\eta}, \quad k = r(r+r_1)^{-1}, \quad r_1 = a/\sigma bl, \tag{3}$$

in which M is the Hartmann number, k is the load factor. and r_1 is the internal resistance of a generator of length l.

In this case, the pressure gradient is a function of time but is uniquely related to the velocity change, the relationship being readily found by integrating (3) over the channel cross section, subject to the condition of constant flow rate:

$$P(t) = -\frac{\nu}{ab}\int_{-1/2b}^{1/2b}\int_{-1/2a}^{1/2a}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)dx\,dy - \frac{M^2\nu}{R_h^2}U(k-1). \tag{4}$$

Then (3) and (4) together give the following integrodifferential equation:

$$\frac{\partial v}{\partial t} = -\frac{\nu}{ab}\int_{-1/2b}^{1/2b}\int_{-1/2a}^{1/2a}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)dx\,dy + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{M^2\nu}{R_h^2}(U - v), \tag{5}$$

* The subscripts to the velocity are omitted here and subsequently.

the boundary conditions for (5) being

$$v\,|_{x=\pm 1/2 b} = v\,|_{y=\pm 1/2 a} = 0\,, \qquad (6)$$

and the initial conditions being given as

$$v = v_0(x, y, 0),$$

$$M = M_0, \quad P = P_0 \quad \text{for} \quad t = 0\,. \qquad (7)$$

The magnetic field is changed instantaneously at the initial instant, and the electromagnetic-pressure loss changes simultaneously, which is physically true because the electromagnetic transients are of very short duration relative to the MHD transients:

$$v = v(x, y, t), \quad M = M,$$

$$P = P_1 + P_2(t) \quad \text{for} \quad t \geqslant 0\,.$$

The solution to (5) is sought in the form

$$v(x, y, t) = F(x, y) + \sum_{n=1}^{\infty} \varphi_n(t)\,\psi_n(x, y)\,, \qquad (8)$$

in which F(x,y) corresponds to the steady-state flow.

Substitution of (8) into (5) gives three differential equations, whose solution is known [4]:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} - \frac{M^2}{R_h^2} F + \frac{M^2}{R_h^2} U -$$

$$- \frac{1}{ab} \int_{-1/2 b}^{1/2 b} \int_{-1/2 a}^{1/2 a} \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} \right) dx\,dy = 0$$

$$\frac{d\varphi_n}{dt} + \alpha_n^2 \varphi_n = 0$$

$$\frac{\partial^2 \psi_n}{\partial x^2} + \frac{\partial^2 \psi_n}{\partial y^2} - \beta_n \psi_n -$$

$$- \frac{1}{ab} \int_{-1/2 b}^{1/2 b} \int_{-1/2 a}^{1/2 a} \left(\frac{\partial^2 \psi_n}{\partial x^2} + \frac{\partial^2 \psi_n}{\partial y^2} \right) dx\,dy = 0\,.$$

The coefficients α_n and β_n are related by

$$\alpha_n^2 = M^2 \nu / R_h^2 - \nu \beta_n\,.$$

The solutions are put as

$$F(x, y) = 4 D_F \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{\lambda_m N_m^2} \left(1 - \frac{\operatorname{ch} N_m x}{\operatorname{ch} N_m {}^1\!/_2 b} \right) \cos \frac{\lambda_m y}{a}\,, \qquad (9)$$

$$\varphi_n(t) = A \exp(-\alpha_n^2 t)\,, \qquad (10)$$

$$\psi_n(x, y) =$$

$$= 4 D_\psi \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{\lambda_m W_{mn}^2} \left(1 - \frac{\operatorname{ch} W_{mn} x}{\operatorname{ch} W_{mn} {}^1\!/_2 b} \right) \cos \frac{\lambda_m y}{a}\,, \qquad (11)$$

$$D_F = \frac{M^2}{R_h^2} U - \frac{1}{ab} \int_{-1/2 b}^{1/2 b} \int_{-1/2 a}^{1/2 a} \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} \right) dx\,dy\,,$$

$$D_\psi = \frac{-1}{ab} \int_{-1/2 b}^{1/2 b} \int_{-1/2 a}^{1/2 a} \left(\frac{\partial^2 \psi_n}{\partial x^2} + \frac{\partial^2 \psi_n}{\partial y^2} \right) dx\,dy\,,$$

$$N_m^2 = M^2 / R_h^2 + \lambda_m^2 / a^2, \quad W_{mn}^2 = \beta_n + \lambda_m^2 / a^2,$$

$$\lambda_m = \pi(2m - 1)\,.$$

It follows from (9) and (11), subject to constancy of flow rate, that

$$ {}^1\!/_2 b W_{mn} = \operatorname{th} W_{mn} {}^1\!/_2 b$$

or

$$ {}^1\!/_2 b\, |W_{mn}| = \operatorname{tg} |W_{mn}|\, {}^1\!/_2 b$$

since $W_{mn}^2 \leq 0$.

We take the Laplace operator of (9) and integrate the result over the cross section of the channel to get the following expression for D_F:

$$D_F = \frac{M^2 \nu U}{R_h^2} \left\{ 1 - 8 \sum_{m=1}^{\infty} \left[\left(\frac{1}{\lambda_m^2} - \frac{1}{a^2 N_m^2} \right) \times \right. \right.$$

$$\left. \left. \times \frac{\operatorname{th} N_m {}^1\!/_2 b}{N_m {}^1\!/_2 b} + \frac{1}{N_m^2 a^2} \right] \right\}^{-1}.$$

The coefficients D_ψ of the series are determined for t = 0 from

$$v_0(x, y) = F_0(x, y) =$$

$$= F(x, y) - 4 \sum_{n=1}^{\infty} D_{1\psi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \cos(\lambda_m y / a)}{\lambda_m\, |W_{mn}^2|} \times$$

$$\times \left[1 - \frac{\cos |W_{mn}|\, x}{\cos |W_{mn}|\, {}^1\!/_2 b} \right],$$

$$F_0(x, y) =$$

$$= 4 D_0 \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \cos(\lambda_m y / a)}{\lambda_m N_{0m}^2} \left[1 - \frac{\operatorname{ch} N_{0m} x}{\operatorname{ch} N_{0m} {}^1\!/_2 b} \right],$$

$$D_0 = \frac{M_0^2}{R_h^2} U - \frac{1}{ab} \int_{-1/2 b}^{1/2 b} \int_{1/2 a}^{1/2 a} \left(\frac{\partial^2 F_0}{\partial x^2} + \frac{\partial^2 F_0}{\partial y^2} \right) dx\,dy\,,$$

$$N_{0m}^2 = \frac{M_0^2}{R_h^2} + \frac{\lambda_m^2}{a^2}, \quad D_{1\psi} = A D_\psi\,. \qquad (12)$$

It follows from (12) that

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{\lambda_m} \cos \frac{\lambda_m y}{a} \left\{ \frac{D_0}{N_{0m}^2} \left[1 - \frac{\operatorname{ch} N_{0m} x}{\operatorname{ch} N_{0m} {}^1\!/_2 b} \right] - \right.$$

$$- \frac{D_F}{N_m^2} \left[1 - \frac{\operatorname{ch} N_m x}{\operatorname{ch} N_m {}^1\!/_2 b} \right] +$$

$$\left. + \sum_{n=1}^{\infty} \frac{D_{1\psi}}{|W_{mn}^2|} \left[1 - \frac{\cos |W_{mn}|\, x}{\cos |W_{mn}|\, {}^1\!/_2 b} \right] \right\} = 0\,.$$

Then

$$\frac{D_0}{N_{0m}^2} \left[1 - \frac{\operatorname{ch} N_{0m} x}{\operatorname{ch} N_{0m} {}^1\!/_2 b} \right] - \frac{D_F}{N_m^2} \left[1 - \frac{\operatorname{ch} N_m x}{\operatorname{ch} N_m {}^1\!/_2 b} \right] =$$

$$= -\sum_{n=1}^{\infty} \frac{D_{1\psi}}{|W_{mn}^2|} \left[1 - \frac{\cos|W_{mn}|x}{\cos|W_{mn}|^1/_2 b}\right].$$

The functions $\varphi_{mn} = 1 - (\cos|W_{mn}|x)/(\cos|W_{mn}|b/2)$ are orthogonal in the range $\pm b/2$, so

$$D_{1\psi} = \frac{D_F}{(^1/_2 bN_m)^3} \frac{N_m b|W_{mn}^2| - 2|W_{mn}^2| \operatorname{th} N_m{}^1/_2 b}{N_m{}^2 + |W_{mn}^2|} -$$

$$- \frac{D_0}{(^1/_2 bN_{0m})^3} \frac{N_{0m} b|W_{mn}^2| - 2|W_{mn}^2| \operatorname{th} N_{0m}{}^1/_2 b}{N_{0m}^2 + |W_{mn}^2|}.$$

This solution allows us, from (4), to calculate the change in the pressure loss in the channel of an MHD generator when the field is switched on ($M_0 = 0$) and when the power output is adjusted via the magnetic induction.

As $t \to \infty$ this solution agrees with the one previously obtained [4] for steady-state flow in a similar channel.

Further, as $M \to 0$ it becomes the known solution for ordinary hydrodynamics.

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